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LETTER TO THE EDITOR

The two-component Kaup–Kupershmidt equation

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Abstract
The Kaup–Kupershmidt equation is generalized to a system of equations in the same manner as the Korteweg–de Vries equation is generalized to the Hirota–Satsuma equation. The Gelfand–Dikii–Lax and Hamiltonian formulation for this generalization is given. The same construction is repeated for the constrained Kadomtsev–Pietviashvili–Lax operator which leads to the four-component Kaup–Kupershmidt equation. The modified version of the two-component Kaup–Kupershmidt equation is presented and analysed.

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1. Introduction

Large classes of nonlinear partial differential equations are integrable by the inverse spectral transform method and its modifications [1, 2]. It is well known that most of the integrable partial differential equations,

$$u_t = F(t, x, u, u_x, u_{xx}, \ldots), \tag{1}$$

admit the so-called Lax representation

$$\frac{\partial L}{\partial t} = [A, L], \tag{2}$$

and hence the inverse-scattering method is applicable.

We shall consider the case where the Lax operator is a differential operator

$$L = \partial^m + u_{m-2}\partial^{m-2} + \cdots + u_0, \tag{3}$$

where u_i, $i = 0, 1, \ldots, m-2$, are functions of x, t. Then equation (2) gives us the Gelfand–Dikii system where $A = L^{n/m}$ is a pseudo-differential series of the form $L^{n/m} = \sum_{-\infty}^{i=n} v_i\partial^i$ and $L^{n/m}_{\geqslant 0} = \sum_{i=0}^{n} v_i\partial^i$.

Quite different systems of equations could be obtained considering the Kadomtsev–Pietviashvili (KP) hierarchy within Sato's approach [3, 4]. In this case, the Lax operator is spanned by infinitely many fields

$$L_{\mathrm{KP}} = \partial + u_1\partial^{-1} + u_2\partial^{-2} + \cdots, \tag{4}$$

with the following Lax pair representation:

$$\frac{\partial L}{\partial t} = [(L^N)_{\geqslant 0}, L]. \tag{5}$$

Both these hierarchies describe large classes of nonlinear partial differential equations. In order to find some interesting equations in these hierarchies, sometimes we need to apply the reduction procedure in which some functions are described in terms of other functions used in the Lax operator. We have no unique prescription how to carry out such a procedure at the moment. Kupershmidt [5] has noted that a certain invariance of the partial differential nonlinear equations can be extracted from the Lax operator. This observation allowed him to put some constraints on the functions appearing in the Lax operator. This procedure is called now the Kupershmidt reduction [1].

In this letter, we would like to consider some specific reduction of the Gelfand–Dikii Lax operator in which the Lax operator can be factorized as the product of two Lax operators. This idea follows from the observation that the product of two Lax operators [6] of the Korteweg–de Vries equations

$$L = (\partial^2 + u)(\partial^2 + v) \tag{6}$$

creates the whole hierarchy of equations with the following Lax pair representation:

$$\frac{\partial L}{\partial t_n} = 8[(L^{(2n+1)/4})_{\geqslant 0}, L], \tag{7}$$

where $n = 0, 1, 2, \ldots$ and the factor 8 was chosen in such a way as to normalize the higher term in the equation. For $n = 1$, we have the Hirota–Satsuma equation [7]

$$\begin{aligned} \frac{\partial u}{\partial t_1} &= (-u_{xxx} + 3v_{xxx} - 6u_x u + 6vu_x + 12v_x u), \\ \frac{\partial v}{\partial t_1} &= (-v_{xxx} + 3u_{xxx} - 6v_x v + 6v_x u + 12vu_x), \end{aligned} \tag{8}$$

while for $n = 2$

$$\begin{aligned} \frac{\partial u}{\partial t_2} = (&-3u_{xxxxx} - 15u_{xxx}u - 15u_{xx}u_x - 15u_x u^2 + 5v_{xxxxx} \\ &+ 25v_{xxx}u + 5v_{xxx}v + 25v_{xx}u_x + 15v_{xx}v_x + 15v_x u_{xx} \\ &+ 20v_x u^2 + 20v_x vu + 5v^2 u_x + 5vu_{xxx} + 30vu_x u)/4, \\ \frac{\partial v}{\partial t_2} = (&5u_{xxxxx} + 5u_{xxx}u + 15u_{xx}u_x - 3v_{xxxxx} + 5v_{xxx}u \\ &- 15v_{xxx}v + 15v_{xx}u_x - 15v_{xx}v_x + 25v_x u_{xx} + 5v_x u^2 \\ &- 15v_x v^2 + 30v_x vu + 20v^2 u_x + 25vu_{xxx} + 20vu_x u)/4. \end{aligned} \tag{9}$$

Let us note that both these equations could be rewritten in the Hamiltonian form as

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_n} = J \begin{pmatrix} \frac{\delta H_n}{\delta u} \\ \frac{\delta H_n}{\delta v} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\partial^3 - 2u\partial - u_x & 0 \\ 0 & -\frac{1}{2}\partial^3 - 2v\partial - v_x \end{pmatrix} \begin{pmatrix} \frac{\delta H_n}{\delta u} \\ \frac{\delta H_n}{\delta v} \end{pmatrix}, \tag{10}$$

where $n = 1, 2$ and

$$H_1 = \int \mathrm{d}x\, \mathrm{Res}(L^{3/2}) = \int \mathrm{d}x\, (u^2 + v^2 - 6uv),$$
$$H_2 = \int \mathrm{d}x\, \mathrm{Res}(L^{5/2}) = \int \mathrm{d}x\, ((3u_{xx} + 10v_{xx})u - u^3 - 3v_{xx}v - v^3 + 5vu(v + u)), \tag{11}$$

and Res denotes the coefficient standing in the ∂^{-1} term.

Recently it was shown in [8] that a similar construction could be carried out for the Harry Dym equation, which leads to the system of interacting equations. However, the Lax operator for the Harry Dym equation does not belong to the Gelfand–Dikii system.

Both these equations could be considered either as the extension of the known equations or as the reduction of the Lax pair representations. Indeed the Lax operator (6) could be considered as the admissible reduction of the fourth-order Gelfand–Dikii–Lax operator

$$L = \partial^4 + f_2\partial^2 + f_1\partial + f_0, \tag{12}$$

where

$$f_2 = u + v, \qquad f_1 = 2v_x, \qquad f_0 = v_{xx} + vu. \tag{13}$$

Now we would like to repeat the similar construction for the Boussinesq-type Lax operators. We choose the third-order Lax operator of the form

$$L = \partial^3 + u\partial + \lambda u_x, \tag{14}$$

where at the moment λ is a free parameter.

This Lax operator generates the whole hierarchy of equations, and the first nontrivial equation starts from the fifth flow

$$\frac{\partial L}{\partial t_5} = 9[(L^{(5/3)})_{\geqslant 0}, L], \tag{15}$$

of the form

$$u_t = \left(-u_{4x} - 5u_{xx}u + 15\lambda(\lambda - 1)u_x^2 - \frac{5}{3}u^3\right)_x \tag{16}$$

only when $\lambda = \frac{1}{2}, 1, 0$. Note that the factor 9 was chosen in such a way as to normalize the higher terms in the equation.

For $\lambda = \frac{1}{2}$, we have the Kaup–Kupershmidt hierarchy [9, 10] while for $\lambda = 1$ or $\lambda = 0$ we obtain the Sawada–Kotera hierarchy [11]. Both these equations are Hamiltonian where

$$u_t = \left(c\partial^3 + \frac{1}{15}(\partial u + u\partial)\right)\frac{\delta H}{\delta u} \tag{17}$$

where

$$H_1 = \int \mathrm{d}x\, \left(3(3\lambda^2 - 3\lambda + 1)u_x^2 - 5u^3\right) \tag{18}$$

and $c = \frac{2}{15}$ for $\lambda = \frac{1}{2}$ or $c = \frac{1}{15}$ for $\lambda = 1$ or $\lambda = 0$.

Now we consider a new Lax operator as the product of two different Lax operators of the Boussinesq type

$$L := (\partial^3 + v\partial + \lambda v_x)(\partial^3 + (u - v)\partial + \lambda(u_x - v_x)). \tag{19}$$

The consistent hierarchy could be obtained only for $\lambda = \frac{1}{2}$, and first two nontrivial flows are

$$\frac{\partial L}{\partial t_n} = 9[(L^{(n/6)})_{\geqslant 0}, L], \tag{20}$$

which give us

$$
\begin{aligned}
v_{t_3} &= \tfrac{9}{2}\left(u_{xxx} - 2v_{xxx} + \tfrac{1}{2}v_x u - 3v_x v + v u_x\right)\\
u_{t_3} &= \tfrac{9}{2}\left(-\tfrac{3}{4}u^2 - 3v^2 + 3uv\right)_x
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
v_{t_5} &= \left(-5u_{xxxx} + 9v_{xxxx} - \tfrac{5}{2}u_{xx}u - \tfrac{5}{2}u_x^2 + 15v_{xx}v + \tfrac{15}{4}v_x^2 + \tfrac{5}{2}v^3 - \tfrac{5}{2}vu_{xx} - \tfrac{5}{8}vu^2\right)_x\\
&\qquad - \tfrac{5}{2}vu_{xxx} - \tfrac{5}{4}vu_x u\\
u_{t_5} &= \left(-u_{xxxx} + 5u_{xx}u + \tfrac{35}{24}u^3 - 15v_{xx}u + 30v_{xx}v + \tfrac{15}{2}v_x^2 - \tfrac{15}{2}v_v u_x + \tfrac{15}{2}u^2 v\right.\\
&\qquad \left. - 15vu_x xx - \tfrac{15}{2}vu^2\right)_x
\end{aligned}
\tag{22}
$$

The last system of the above equations is our two-component generalized Kaup–Kupershmidt equation. This system cannot be reduced to the system of equations (9) by the linear transformation.

Interestingly this two-component generalization has been considered for the first time in [4] where the authors investigated the so-called constrained Kadomtsev–Pietviashvilli hirerarchy. The constrained KP hierarchy is obtained from the usual KP hierarchy as

$$
L_{\mathrm{KP}}^N = \left(L_{\mathrm{KP}}^N\right)_{\geqslant 0} + \Psi\partial^{-1}\Phi,
\tag{23}
$$

with L_{KP} defined by (4). Then equations (21) and (22) could be obtained by choosing

$$
L_{\mathrm{KP}} = \partial^3 + \tfrac{1}{2}u\partial + \tfrac{1}{4}u_x + \tfrac{1}{16}(2v-u)\partial^{-1}(2v-u).
\tag{24}
$$

In contrast to the usual Kaup–Kupershmidt hierarchy, which starts from the fifth flow, our hierarchy begins from the third flow. Note that our Lax operator as well as the equations allows the reduction to the standard Kaup–Kupershmidt Lax operator or equations when $u = 2v$.

Both these systems are Hamiltonian where

$$
\begin{pmatrix} u\\ v\end{pmatrix}_{t_n} = J\frac{\delta H_n}{\delta v} = \frac{1}{216}\begin{pmatrix} 4\partial^3 + \partial u + u\partial & 2\partial^3 + \partial v + v\partial\\ 2\partial^3 + \partial v + v\partial & 2\partial^3 + \partial v + v\partial\end{pmatrix}\begin{pmatrix}\frac{\delta H_n}{\delta u}\\ \frac{\delta H_n}{\delta v}\end{pmatrix}
\tag{25}
$$

and

$$
\begin{aligned}
H_3 &= \int \mathrm{d}x\,\mathrm{Res}(L^{3/6}) = 54\int \mathrm{d}x\,(4uv - 4v^2 - u^2)\\
H_5 &= \int \mathrm{d}x\,\mathrm{Res}(L^{5/6}) = \int \mathrm{d}x\,(7u^3 + 24u_{xx}u - (108v_{xx} - 36vu)(v-u)).
\end{aligned}
\tag{26}
$$

By straightforward calculations, it is easy to show that the Hamiltonian operator J satisfies the Jacobi identity.

Let us now consider the following Miura transformation:

$$
u = a_x, \qquad v = b_x - \tfrac{1}{4}b^2,
\tag{27}
$$

where a, b are functions of x and t. It is easy to show that this transforms the system of equations

$$
\begin{aligned}
a_{t_3} &= \tfrac{1}{16}\left(-12a_x^2 - 48b_x^2 + 48b_x a_x + 24b_x b^2 - 3b^4 - 12b^2 a_x\right)\\
b_{t_3} &= \tfrac{1}{4}(4a_{xx} - 8b_{xx} + b^3 + 2ba_x)_x
\end{aligned}
\tag{28}
$$

$$a_{t_5} = \tfrac{1}{96}\big(-96a_{xxxxx} + 480a_{xxx}a_x + 140a_x^3 + 1440b_{xxx}(2b_x - a_x) + 720b_{xx}\left(-a_{xx} - 3b_xb + \tfrac{1}{2}b^3\right) + 45b^4a_x + 360b^2a_{xxx} + 180b^2a_x^2 + 360b_x\left(-4b_x^2 + 4b_xa_x + \tfrac{3}{2}b_xb^2 - 4a_{xxx} - b^2a_x + ba_{xx}\right)\big) \tag{29}$$

$$b_{t_5} = \tfrac{1}{32}\big(-160a_{xxxx} - 80a_{xx}a_x + 288b_{xxxx} - 240b_{xx}b_x - 120b_{xx}b^2 - 12b_x^2b + 3b^5 - 80ba_{xxx} - 20ba_x^2\big)_x$$

to the system (21) or (22) respectively.

Note that equations (28) describe the system of two interacting fields of the modified Korteweg–de Vries type. This system of equations does not belong to the class of interacting fields considered by Foursov [12]. Foursov has classified all integrable systems of two interacting modified KdV-type equations which could be reduced to the symmetrical form

$$u_t = F[u, v], \qquad v_t = F[v, u], \tag{30}$$

where $F[u, v] = F[u, u_x, u_{xx}, \ldots, v, v_x, v_{xx}, \ldots]$ denotes differential polynomial function of two variables. However, our system of equations (24) cannot be reduced to the symmetrical form by the linear transformation.

Interestingly system (28) collapses when $u = 2v$. Indeed the condition $u = 2v$ is equivalent with the assumption that

$$a_x = 2b_x - \tfrac{1}{2}b^2, \tag{31}$$

and therefore we have $a_{t_3} = 0$. The system of equation (29) reduces when $u = 2v$ to the modified version of the Kaup–Kupershmidt equation

$$b_t = \tfrac{1}{16}\big(-16b_{xxxx} - 40b_{xx}b_x + 20b_{xx}b^2 + 20b_x^2b - b^5\big)_x. \tag{32}$$

Our equations (28) and (29) are Hamiltonian equations where

$$\begin{pmatrix} a \\ b \end{pmatrix}_{t_n} = \mathcal{D}\begin{pmatrix} \frac{\delta H_n}{\delta a} \\ \frac{\delta H_n}{\delta b} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -4\partial - (\partial^{-1}a_x - a_x\partial^{-1}) & -2\partial - \partial^{-1}b_x + b \\ -2\partial - b - b_x\partial^{-1} & -2\partial \end{pmatrix}\begin{pmatrix} \frac{\delta H_n}{\delta a} \\ \frac{\delta H_n}{\delta b} \end{pmatrix}, \tag{33}$$

where $n = 3, 5$ and

$$H_3 = \int \mathrm{d}x\left(\frac{1}{2}a_{xx}a - 2b_{xx}a + 2b_{xx}b + b_xba - \frac{1}{8}b^4\right)$$

$$H_5 = \int \mathrm{d}x(24a_{xxxx}a + 14a_{xx}a_xa - 108b_{xxxx}(a - b) + 54b_{xxx}b_xb + b_{xx}(234b_xa - 36a_xa - 108b_xb - 18b^2a) + b_x^2(27b^2 - 36ba) + b_x(bb^3a - 36a_{xx}a + 18ba_aa) + 9b^2a_{xx}a). \tag{34}$$

It is easy to check that the operator $\mathcal{D}$ is the Hamiltonian operator. Indeed it is enough to note that under the Miura transformation (27) this operator transforms to the $\hat{J} = \mathcal{F}\mathcal{D}\mathcal{F}^\star$, where $\mathcal{F}$ is the Freche derivative of the Miura transformation and $\star$ denotes the Hermitian conjugation.

$$\hat{J} = \begin{pmatrix} \partial & 0 \\ 0 & -\partial - \frac{1}{2}b \end{pmatrix}. \tag{35}$$

Let us apply finally the factorization procedure directly to the constrained Kadomtsev–Petviashvili–Lax operator. We consider therefore two different Lax operators

$$L_1 = \partial^3 + v\partial + \tfrac{1}{2}v_x + h\partial^{-1}h, \qquad L_2 = \partial^3 + (u - v)\partial + \tfrac{1}{2}(u_x - v_x) + g\partial^{-1}g, \tag{36}$$

and construct the new Lax operator as

$$L = L_1 L_2. \tag{37}$$

This Lax operator generates the integrable hierarchy of four interacting fields. The first nontrivial equations are

$$\begin{aligned}
u_{t_3} &= \tfrac{9}{2}\left(6g^2 + 6h^2 - \tfrac{3}{2}u^2 + 6vu - 6v^2\right)_x \\
v_{t_3} &= \tfrac{9}{2}(12hh_x + 2u_{xxx} - 4v_{xxx} + v_x u - 6v_x v + 2vu_x) \\
g_{t_3} &= \tfrac{9}{2}(2g_{xxx} - u_x g + ug_x + 3v_x g) \\
h_{t_3} &= \tfrac{9}{2}(2h_{xxx} + 2u_x h + uh_x - 3v_x h),
\end{aligned} \tag{38}$$

$$\begin{aligned}
u_{t_5} &= \big(60g_{xx}g + 15g_x^2 + 60h_{xx}h + 15h_x^2 - u_{xxxx} + 5u_{xx}u + \tfrac{35}{24}u^3 - \tfrac{15}{2}ug^2 + \tfrac{75}{2}uh^2 \\
&\qquad - 15v_{xx}u + 30v_{xx}v + \tfrac{15}{2}v_x^2 - \tfrac{15}{2}v_x u_x + \tfrac{15}{2}v^2 u + 45vg^2 \\
&\qquad - 45vh^2 - 15vu_{xx} - \tfrac{15}{2}vu^2\big)_x, \\
v_{t_5} &= 30g_{xxx}g + 90g_{xx}g_x + 30h_{xxx}h - 5u_{xxxxx} - \tfrac{5}{2}u_{xxx}u - \tfrac{15}{2}u_{xx}u_x + 30u_x h^2 + 45uh_x h \\
&\qquad + 9v_{xxxxx} + 15v_{xxx}v + \tfrac{45}{2}v_{xx}v_x + \tfrac{15}{2}v_x g^2 - \tfrac{75}{2}v_x h^2 - \tfrac{5}{2}v_x u_{xx} - \tfrac{5}{8}v_x u^2 \\
&\qquad + \tfrac{15}{2}60v_x v^2 + 30vg_x g - 60vh_x h - 5vu_{xxx} - \tfrac{5}{2}vu_x u \\
g_{t_5} &= 9g_{xxxxx} - \tfrac{15}{2}g_x g^2 + 30h_x hg + \tfrac{15}{2}h^2 g_x + \tfrac{5}{2}u_{xxx}g - \tfrac{5}{2}u_{xx}g_x + \tfrac{5}{4}u_x ug - \tfrac{5}{8}u^2 g_x \\
&\qquad + \tfrac{15}{2}ug_{xxx} + \tfrac{45}{2}v_{xx}g_x + \tfrac{45}{2}v_x g_{xx} + \tfrac{15}{4}v_x ug - \tfrac{15}{2}v_x vg - \tfrac{15}{2}v^2 g_x + \tfrac{15}{2}vug_x \\
h_{t_5} &= 9h_{xxxxx} + \tfrac{15}{2}h_x g^2 - \tfrac{15}{2}h_x h^2 + 30hg_x g + \tfrac{5}{2}u_{xxx}h + 20u_{xx}h_x + \tfrac{45}{2}u_x h_{xx} \\
&\qquad - \tfrac{5}{2}u_x uh - \tfrac{5}{8}u^2 h_x + \tfrac{15}{2}uh_{xxx} - \tfrac{45}{2}v_{xx}h_x - \tfrac{45}{2}v_x h_{xx} \\
&\qquad + \tfrac{15}{4}v_x uh - \tfrac{15}{2}v_x vh - \tfrac{15}{2}v^2 h_x + \tfrac{15}{2}vu_x h + \tfrac{15}{2}vuh_x.
\end{aligned} \tag{39}$$

The last system of equations could be considered as the four-component generalized Kaup–Kupershmidt equation. This equation reduces to the two-component Kaup–Kupershmidt equation when $g = h = 0$.

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